Fatou Sets for Holomorphic Surface Automorphisms

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In the dynamics of holomorphic surface diffeomorphisms, there are two basic cases: Dissipative (volume decreasing) and Conservative (volume preserving). The Fatou set is the region is where the dynamical behavior is well-behaved/regular. Invariant Fatou components, in many cases, are biholomorphically equivalent to Reinhardt domains.

If you see a Reinhardt domain "in nature", how would you recognize it?

First of two classes of maps we consider:

1. Nontrivial elements of $f \in PolyAut(\mathbb{C}^2)$:

$$\mathrm{ddeg}(f) := \lim_{n\to\infty} (\deg(f^n))^{1/n} > 1$$

If $\mathrm{ddeg}(f) = 2$, then (up to conjugacy) we have a *Hénon map*

$$f(x, y) = (x^2 + c - ay, x), \quad f^{-1}(x, y) = \left(y, \frac{y^2 + c - x)}{a}\right)$$

$$K^{+/-} := \{(x, y) \in \mathbb{C}^2 : \{f^{\pm n}(x, y) : n \geq 0\} \text{ is bounded } \}$$

$$K := K^+ \cap K^-$$

Julia sets: $J^\pm = \partial K^\pm, \quad J := J^+ \cap J^-$
Cases:

- $|a| < 1$, volume-decreasing, dissipative
- $|a| = 1$, volume-preserving, conservative

Fatou set: regular behavior

$\mathcal{F}^+ := \{p \in \mathbb{C}^2 : U \ni p : \{f^n|_U : n \geq 0\}$ is a normal family $\}$
For Hénon maps, we have

$$\mathcal{F}^+ = \mathrm{int}(K^+) \cup (\mathbb{C}^2 - K^+)$$

Theorem (B-Smillie)

Suppose that f is dissipative, and $f(\Omega) = \Omega$ is an invariant Fatou component satisfying $\{f^n(p) : n \geq 0\} \Subset \Omega$ for some $p \in \Omega$. Then one of the following occurs:

1. $\Omega =$ *basin of attracting fixed point, and* $\Omega \cong \mathbb{C}^2$.
2. $\Omega =$ *basin of a rotational disk, and* $\Omega \cong \mathbb{C}^2$.
3. $\Omega =$ *basin of a rotational annulus, and* $\Omega \cong A \times \mathbb{C}$.

Open Question: Can case #3 occur?

Fatou components whose orbits are non-recurrent

$(0,0)$ is semi-parabolic if $f(0,0) = (0,0)$, and $Df(0,0) \cong \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ The *basin* $\mathcal{B}$ of $(0,0)$ is the set of all points which converge locally uniformly to $(0,0)$. In particular, $(0,0) \notin \mathcal{B}$.

Theorem (Ueda, Hakim)

For a semi-parabolic/semi-attracting basin, we have $\mathcal{B} \cong \mathbb{C}^2$.

Theorem (Lyubich-Peters)

Suppose that f is dissipative, $|a| < 1/4$, and $f(\Omega) = \Omega$ is an invariant Fatou component for which there is no $p \in \Omega$ such that $\{f^n(p) : n \geq 0\} \Subset \Omega$. Then Ω is a semi-parabolic basin of a semi-parabolic fixed point

Open Question: Is the hypothesis $|a| < 1/4$ necessary?

Existence of Fatou components

We can construct Fatou components by local linearization. Suppose that $f(0,0) = (0,0)$, and $Df(0,0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. We say that $(0,0)$ is non-resonant if there are no $n_1, n_2 \geq 0$, $n_1 + n_2 \geq 2$ such that $\lambda_1 = \lambda_1^{n_1}\lambda_2^{n_2}$ or $\lambda_2 = \lambda_1^{n_1}\lambda_2^{n_2}$. Then there is a formal power series

$$\Phi(z_1, z_2) = \sum a_{n_1,n_2} z_1^{n_1} z_2^{n_2}$$

such that

$$L \circ \Phi = \Phi \circ f, \text{ where } L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Theorem (C.L. Siegel, “small divisors”)

If the pair λ_1, λ_2 is “sufficiently irrational”, then the power series Φ converges in a neighborhood of $(0,0)$.

We may choose $|\lambda_1| = 1 > |\lambda_2|$ to have a Fatou components satisfying #2 above. We may also choose $|\lambda_1| = |\lambda_2| = 1$.

Conservative maps

Theorem (Friedland-Milnor)

If $|a| = 1$ (conservative case), every Fatou component Ω is bounded and satisfies $f^N(\Omega) = \Omega$ for some $N \neq 0$.

Theorem

Suppose that $(0,0) \in \Omega$ is a fixed point. Then the sequence

$$\Phi(p) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} L^{-j} \circ f^j(p) \ : \Omega \to \mathbb{C}^2$$

satisfies $\Phi \circ f = L \circ \Phi$.

Thus every Fatou component with a fixed point can be linearized, and thus could have been constructed as in Siegel's Theorem.

Every Fatou component is a rotation domain

$\Omega = \mathcal{F}$-component. We replace f by f^N so that we have $f(\Omega) = \Omega$.

$$\mathcal{G} := \{\text{limits } g := \lim_{j\to\infty} f^{n_j} \ : \Omega \to \Omega\}$$

$\mathcal{G}$ is a compact Lie group; $\mathcal{G}_0 \cong \mathbb{T}^\rho$ (real torus of dimension ρ). We say that Ω is a *rotation domain*, and $\rho = \rho(\Omega)$ is its *rank*.

Theorem (B-Smillie)

Only possibilities are $\rho = 1, 2$.

What we will observe: If Ω is a rotation domain of rank ρ, then for every point $p_0 \in \Omega$, the closure of the orbit $\{f^n(p_0) : n \in \mathbb{N}\}$ (or, equivalently, $\{f^n(p_0) : n \in \mathbb{Z}\}$), will be a real analytic (smooth) ρ-torus.

Uniformization – Which Reinhardt domain?

Theorem (Barrett-B-Dadok)

If $\rho(\Omega) = 2$, then there is a Reinhardt domain $\mathcal{D}$ and a biholomorphic conjugacy $\Phi : (\Omega, f) \to (\mathcal{D}, L)$.

Comment: $\mathcal{D}$ is the domain of convergence for the power (or Laurent) series for the map $\Phi^{-1} : \mathcal{D} \to \Omega$.

Question: What are the possibilities for $\mathcal{D}$? (Nothing known.)
Question: Does Ω (necessarily) contain a fixed point? $\Leftrightarrow$ Is $(0,0) \in \mathcal{D}$?

Comment: If Ω has no fixed point, then it cannot be constructed using Siegel's (Linearization) Theorem. This is why we are interested in the existence of such Fatou components.

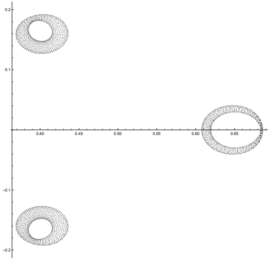
What do we expect to see when we plot an orbit?

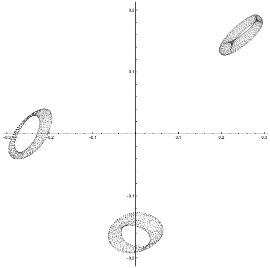
Ushiki Example: Orbits near a rotational 3-cycle

Ushiki's example is an area-preserving Hénon map.

$$f(x,y) = (e^{i\theta}(x^2+\alpha) - e^{2i\theta}y, x), \quad \theta = 1.02773,\ \alpha = 0.269423$$

These orbits look like "reasonable" tori if the starting point is not too far from the fixed point:

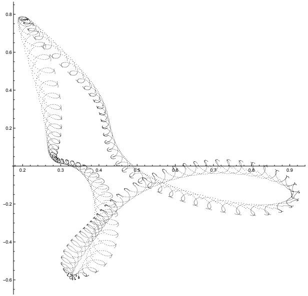


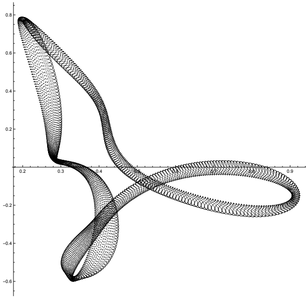


Ushiki's Example, continued: Exotic rotation domain

Same map as before: this is quadratic, so it has 2 fixed points, and both are of saddle type. Therefore, a Fatou component with $f(\Omega) = \Omega$ cannot contain a fixed point.

Two pictures of the same orbit. We see how things fill out as we go from 5000 to 50000 points. We also see that it appears to be: (1) rank 2, and (2) connected.

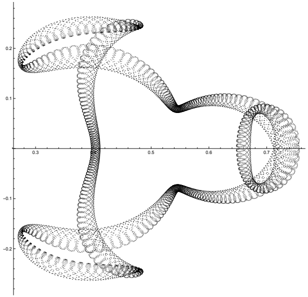


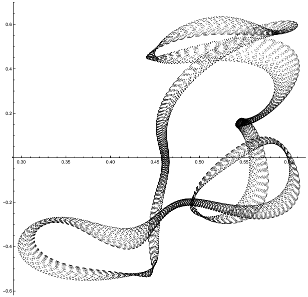


A Second Exotic Rotation Domain in Ushiki's Example

Same map as before: this is quadratic, so it has 2 fixed points, and both are of saddle type. Therefore, a Fatou component with $f(\Omega) = \Omega$ cannot contain a fixed point.

Two different projections of 40000 points from the same orbit. We see that it appears to be: (1) rank 2, and (2) connected.





The pictures are suggestive, and these parameters can be "jiggled": *Is it possible to prove that such exotic rotation domains actually exist?*

Automorphisms of Compact, Complex Surfaces

There are many parallels between the dynamics of complex Hénon maps and the dynamics of automorphisms of compact, complex surfaces.

Theorem (B-Kyounghee Kim)

Let f be a volume-preserving automorphsm of a compact, complex surface X, and suppose that f has positive entropy. Then every Fatou component satisfying $f(\Omega) = \Omega$ is a rotation domain of rank 1 or 2.

Let us consider the family of maps:

$$f_{a,b}(x, y) = \left(y, \frac{y + a}{x + b}\right)$$

Theorem (B-Kyounghee Kim)

There is a blowup $\pi : X \to \mathbb{P}^2$ at $n + 3$ points such that $f_{a,b}$ lifts to an automorphism of X if and only if $f^n_{a,b}(-a, 0) = (-b, -a)$. The dynamical degree of $f_{a,b}$ in this case is λ_n. If $n \geq 7$, then $\lambda_n > 1$.

Some of the automorphisms given in this Theorem have invariant curves, and some do not. The curve $\{y = x^3\}$ is a cubic with a cusp at infinity. We let $\mathcal{C}$ denote the image of this cubic under a linear automorphism of $\mathbb{P}^2$, and we let $\eta = dx \wedge dy/p(x, y)$ denote the 2-form with simple pole along $\mathcal{C}$. We interpret η as a singular volume form.

Theorem (McMullen)

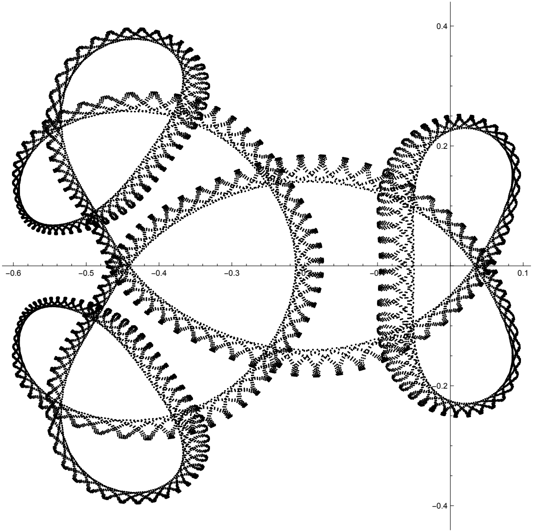
Let $\lambda > 0$ be as above. Let δ be a Galois conjugate of λ, other than $1/\lambda$. Then $|\delta| = 1$, and there is an automorphism $f_{a,b}$ satisfying

$$f^*\eta = \delta\eta$$

When $n = 7$, the number λ is an algebraic number of degree 10. So there are 5 pairs of Galois conjugates δ, δ^{-1}. Ushiki explored these maps and found that one of them has an exotic rotation domain.

Ushiki's Discovery: Automorphism with Exotic Rotation

For $n = 7$, one of the 4 maps with $|\delta| = 1$ has a point whose orbit is as pictured. There are no points that can be centers of rotation for this domain, so it must be "exotic":



My Favorite Imbedding of a Torus

Theorem (B-Kyounghee Kim)

Under certain conditions on c and β, there is a blowup $\pi : X \to \mathbb{P}^2$ such that the map

$$g_{c,\beta}(x, y) = (y, \beta(cy + 1/y) - \beta^2 x)$$

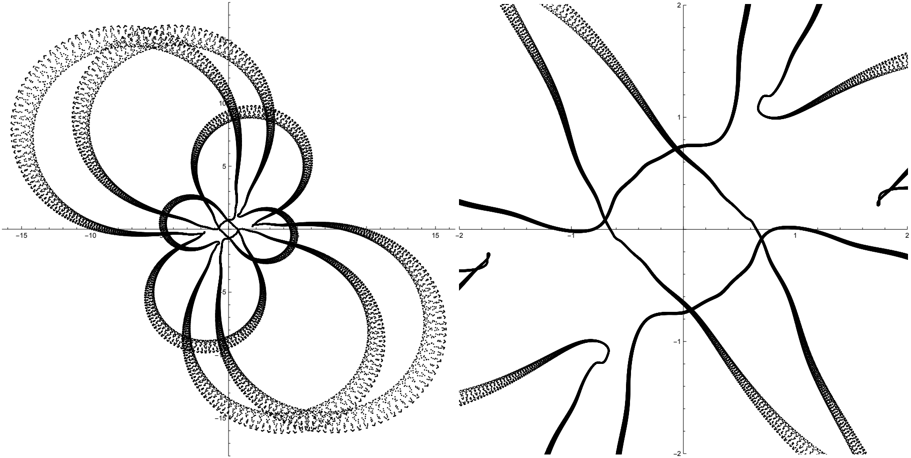
induces a rational surface automorphism g_X.

We also have an analogue of the Theorem with Barrett and Dadok:

Theorem (B-Kyounghee Kim)

Let $g_{c,\beta}$ be a map as above which is volume preserving. Then the line at infinity is invariant and contained in a rotation domain of rank 1. There is a toric blowup $\pi : M \to \mathbb{P}^2$ and an t imbedding $\Phi : \Omega \to \mathcal{D} \subset M$ such that $L \circ \Phi = \Phi \circ f$. Further, the domain $\mathcal{D}$ is pseudoconvex.

For certain c and β as in the theorem above, a $g_{c,\beta}$ orbit of length 220000. There is no fixed point for this domain, so it must be "exotic". Orbit on left; detail on right, to show that orbit closure must be connected:



Real Surface Automorphisms

It is an interesting open problem to determine all the rational surface automorphsms. By a Theorem of Nagata, every rational surface automorphism may be obtained from a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, and then lifting to a blowup $\pi : X \to \mathbb{P}^2$. An interesting starting place is to consider birational maps of degree 2. The nondegenerate prototype is the Cremona involution:

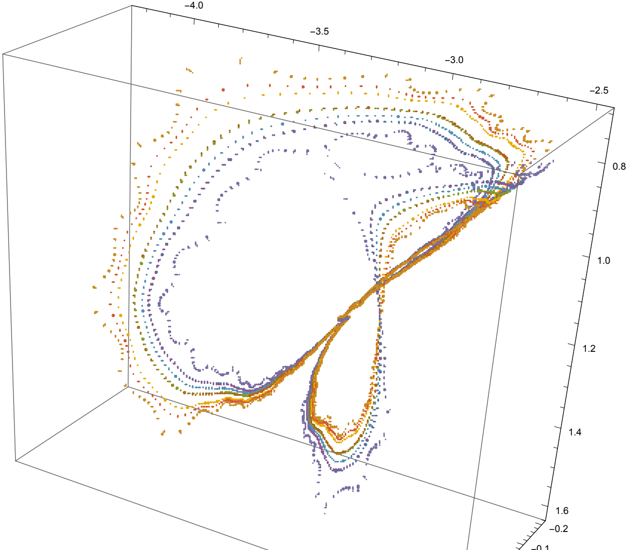
$$J(x : y : z :) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad J(x : y : z) = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$$

Birational mappings of degree 2 may be conjugated to the form $L \circ J$, where $L \in PGL(3, \mathbb{C})$. A recent paper of Diller and Kim has explored the possibilities for automorphisms in the case where $L \in PGL(3, \mathbb{R})$. In this case, the map $L \circ J$ induces both a complex automorphism on the blowup space X, but also a diffeomorphism of the real points $X_{\mathbb{R}}$. These maps are determined by *orbit data* $(n_0, n_1, n_1), \sigma$, where $n_j \geq 0$, and σ is a permutation of $\{0, 1, 2\}$.

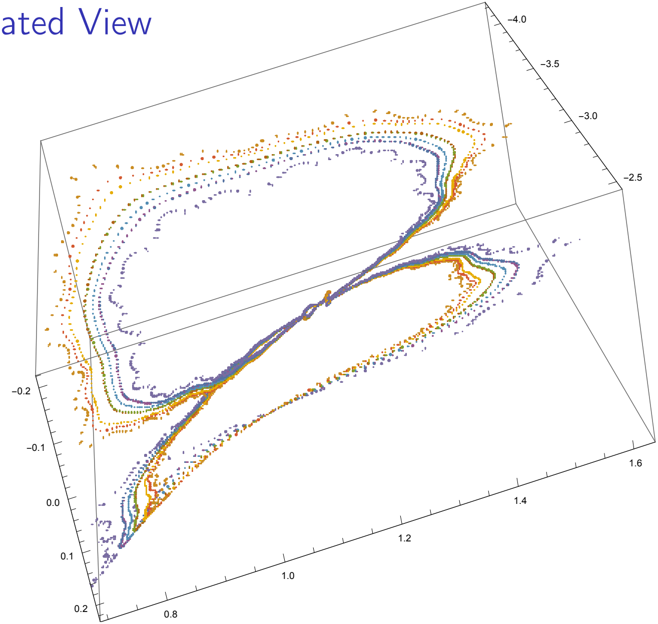
Ushiki found a possible Attractng Herman ring

Ushiki considered the real (volume contracting) map corresponding to the data $(3, 3, 4)$, $\sigma = (1, 2, 0)$ (cyclic). Orbits of 9 random points give the following picture:

Out[∘]=



Rotated View



Summary

We have seen a number of "computer phenomena", mostly due to S. Ushiki. Can we turn this into mathematics?

- In the case of volume-preserving complex maps: can you identify something about what Reinhardt domains can (or cannot) appear as linearizations?
- In the case of rational surface maps, can you show that exotic rotation domains actually exist?
- Can you prove the existence of an attracting Herman ring?